

On the empirical spectral distribution for matrices with long memory and independent rows

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Abstract

In this paper we show that the empirical eigenvalue distribution of any sample covariance matrix generated by independent copies of a stationary regular sequence has a limiting distribution depending only on the spectral density of the sequence. We characterize this limit in terms of Stieltjes transform via a certain simple equation. No rate of convergence to zero of the covariances is imposed. If the entries of the stationary sequence are functions of independent random variables the result holds without any other additional assumptions.

As a method of proof, we study the empirical eigenvalue distribution for a symmetric matrix with independent rows below the diagonal; the entries satisfy a Lindeberg-type condition along with mixingale-type conditions without rates. In this nonstationary setting we point out a property of universality, meaning that, for large matrix size, the empirical eigenvalue distribution depends only on the covariance structure of the sequence and is independent on the distribution leading to it. These results have interest in themselves, allowing to study symmetric random matrices generated by random processes with both short and long memory.

1 Introduction and the main Result.

Due to the fact that random matrices appear in many applied fields, their empirical spectral distribution is a subject of intense research. Earlier works, pioneered by the celebrated paper by Wigner (1958), deal with symmetric matrices having independent entries below the diagonal. Only in the last two decades there has been an effort to weaken the hypotheses of independence and various forms of weak dependence have been considered. The progress was in general achieved first for Gaussian random matrices. For this case the joint distribution of eigenvalues is tractable. Among the papers for symmetric Gaussian matrices with correlated entries we mention the works of Khorunzhy and Pastur (1994), Boutet de Monvel *et al.* (1996), Boutet de Monvel and Khorunzhy (1999), Chakrabarty *et al.* (2014).

Our paper is essentially motivated by the study of large sample covariance matrices, which is a very important topic in multivariate analysis. The spectral analysis of large-dimensional sample covariance matrices has been actively studied starting with the work of Marčenko and Pastur (1967). Extensions can be found in the works of Wachter (1978), Yin (1986), Silverstein (1995), Silverstein and Bai (1995), Hachem *et al.* (2005), Bai and Zhou (2008), Adamczak (2011), Pfaffel and Schlemm (2011), Yao (2012), Banna and Merlevède (2013).

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In this paper, in Theorem 1, we find the limiting empirical eigenvalue distribution for the sample covariance matrix of a stationary process which is regular. The regularity is an ergodic-type property. Our result shows that the limit can be obtained much beyond the situation of weakly dependent case which corresponds to continuous and bounded spectral densities, short range dependence and absolutely summable covariances. It applies to long range dependent stationary stochastic processes and sheds light on the theory of sample covariance random matrices, which is important in large sample statistics for stochastic processes. We show that the limit of the empirical spectral distribution exists and we also characterize the limit in terms of its Stieltjes transform, even for the case when the spectral density is not continuous or even square integrable. The previous works on short memory processes heavily relied upon the limits of Toeplitz matrices induced by the covariance structure. In our case, the covariance matrices fall outside the range of the celebrated Szegő-Trotter theorem for Toeplitz matrices, which is restricted to square summable entries, therefore to square integrable spectral densities. We showed that the spectral density of the underlying stationary process is a key factor to describe the limit in all the situations.

Furthermore, the technical theorems leading to Theorem 1 are also important. They reduce the study of the empirical spectral distribution of symmetric matrices with independent regular rows, below diagonal, to the study of the sequence of the expected value of Stieltjes transforms associated to a Gaussian matrix with the same covariance structure. These results are set in the non-stationary case for variables satisfying a certain Lindeberg condition. Their proofs are complicated by the fact that our intention was to avoid the use of rates of decay of the covariances.

In order to stress the importance of our results we include several applications to regular processes, functions of i.i.d., and linear processes with martingale differences innovations. As we shall see, Theorem 1 applies to large sample covariance matrices constructed from independent copies of any stationary process whose entries are functions of i.i.d. which are centered and has finite second moments. In particular the theorem applies to any causal linear process with square summable coefficients and i.i.d. innovations as soon as the process exists in \mathbb{L}^2 , so it could have long memory.

Our proofs are a blend of probabilistic techniques for dependent structures such as the big and small block argument and martingale approximations, properties of Gaussian processes, and algebraic and Fourier analysis tools. Because our variables are correlated the method of proof is based on the Stieltjes transform, which is well adapted to handle dependent entries. The Stieltjes transform is also useful to characterize the limit.

Here are some notations used all along the paper. The notation $[x]$ is used to denote the integer part of a real x . The notation $\mathbf{0}_p$ means a row vector of size p with components equal to zero. When no confusion is possible concerning the size of a null vector $\mathbf{0}$ we will omit the index of its size. For a matrix A , we denote by A^T its transpose matrix, by $\text{Tr}(A)$ its trace. We shall also use the notation $\|X\|_r$ for the \mathbb{L}^r -norm ($r \geq 1$) of a real valued random variable X .

For any sequence of square matrices A_n of order n with only real eigenvalues $\lambda_{1,n} \leq \dots \leq \lambda_{n,n}$, the spectral distribution function is defined by

$$F^{A_n}(x) = \frac{1}{n} \sum_{k=1}^n I(\lambda_{k,n} \leq x),$$

where $I(B)$ denotes the indicator of an event B . The general problem is to find a distribution function F such that $F^{A_n} \rightarrow F$ at all points of continuity of F , or equivalently $d(F^{A_n}, F) \rightarrow 0$, where the Lévy distance between two distribution functions F and G is defined by

$$d(F, G) = \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon\}.$$

The Stieltjes transform of F^{A_n} is given by

$$S^{A_n}(z) = \int \frac{1}{x-z} dF^{A_n}(x) = \frac{1}{n} \text{Tr}(A_n - z\mathbf{I}_n)^{-1},$$

where $z = u + iv \in \mathbb{C}^+$ (the set of complex numbers with positive imaginary part), and \mathbf{I}_n is the identity matrix of order n . It is well-known that $\lim_{n \rightarrow \infty} d(F^{A_n}, F) = 0$ if and only if for all $z \in \mathbb{C}^+$, $S_{A_n}(z) \rightarrow S_F(z)$. We can also see, for instance, in Proposition 2.1 in Bobkov *et al.* (2010), that the estimate of the Lévy distance between empirical spectral distribution functions associated with two matrices can be also given in terms of their Stieltjes transforms.

Let N and p be two positive integers and consider the $N \times p$ matrix

$$\mathcal{X}_{N,p} = (X_{ij})_{1 \leq i \leq N, 1 \leq j \leq p}, \quad (1)$$

where X_{ij} 's are real-valued random variables. Define now the symmetric matrix \mathbb{B}_N of order p by

$$\mathbb{B}_N = \frac{1}{N} \mathcal{X}_{N,p}^T \mathcal{X}_{N,p}. \quad (2)$$

The matrix \mathbb{B}_N is usually referred to as the sample covariance matrix associated with the process $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$. It is also known under the name of Gram random matrix.

In Theorem 1 below, we consider N independent copies $(X_{ij})_{j \in \mathbb{Z}}, i = 1, \dots, N$ of a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ of real-valued random variables in \mathbb{L}^2 and give sufficient conditions to characterize the limiting distribution of $F^{\mathbb{B}_N}$ when $p/N \rightarrow c \in (0, \infty)$. Relevant to this characterization is the notion of spectral distribution function induced by the covariances of $(X_i)_{i \in \mathbb{Z}}$. By Herglotz's Theorem (see e.g. Brockwell and Davis [12]), there exists a non-decreasing function G (the spectral distribution function) on $[-\pi, \pi]$ such that, for all $j \in \mathbb{Z}$, $\text{Cov}(X_0, X_j) = \int_{-\pi}^{\pi} \exp(ij\theta) dG(\theta)$. If G is absolutely continuous with respect to the normalized Lebesgue measure λ on $[-\pi, \pi]$, then the Radon-Nikodym derivative f of G with respect to the Lebesgue measure is called the spectral density, it is a nonnegative, even and integrable function on $[-\pi, \pi]$ which satisfies

$$\text{Cov}(X_0, X_j) = \int_{-\pi}^{\pi} \exp(ij\theta) f(\theta) d\theta, \quad j \in \mathbb{Z}.$$

We shall introduce the following regularity conditions. Define the left tail sigma field of $(X_i)_{i \in \mathbb{Z}}$ by $\mathcal{G}_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{G}_k$ where $\mathcal{G}_k = \sigma(X_j, j \leq k)$

$$\mathbb{E}(X_0 | \mathcal{G}_{-\infty}) = 0 \text{ a.s.} \quad (3)$$

and for every integer k

$$\mathbb{E}(X_0 X_k | \mathcal{G}_{-\infty}) = \mathbb{E}(X_0 X_k) \text{ a.s.} \quad (4)$$

We point out that if (3) holds, then the process $(X_k)_{k \in \mathbb{Z}}$ is purely non deterministic. Hence, by a result of Szegő (see for instance [6, Theorem 3]) if (3) holds, the spectral density f of $(X_k)_{k \in \mathbb{Z}}$ exists and if X_0 is non degenerate,

$$\int_{-\pi}^{\pi} \log f(t) dt > -\infty;$$

in particular, f cannot vanish on a set of positive measure.

Theorem 1. *Consider N independent copies $(X_{ij})_{j \in \mathbb{Z}}, i = 1, \dots, N$ of a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ of real-valued random variables centered and in \mathbb{L}^2 and that satisfies the conditions (3) and (4). Assume $p/N \rightarrow c \in (0, \infty)$. Then there is a nonrandom probability distribution F such that $d(F^{\mathbb{B}_N}, F) \rightarrow 0$ a.s. Furthermore, the Stieltjes transform $S = S(z), z \in \mathbb{C}^+$, of F is determined by the equation*

$$z = -\frac{1}{\underline{S}} + \frac{c}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\underline{S} + (2\pi f(\lambda))^{-1}} d\lambda, \quad (5)$$

where $\underline{S} := -(1-c)/z + cS$ and $f(\cdot)$ is the spectral density of $(X_k)_{k \in \mathbb{Z}}$.

Remark 2. As a matter of fact, we can relax the stationarity to stationarity in \mathbb{L}^2 . More precisely, the conclusion of Theorem 1 applies for Gram matrices constructed from a process $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$ satisfying the conditions of Theorem 5 below if we assume in addition that for any i, k, ℓ in \mathbb{Z}

$$\text{Cov}(X_{ik}, X_{i\ell}) = \text{Cov}(X_{0k}, X_{0\ell}) = \text{Cov}(X_{00}, X_{0, \ell-k}).$$

In this case, $f(\cdot)$ is the spectral density of $(X_{0k})_{k \in \mathbb{Z}}$.

Note that if $\mathcal{G}_{-\infty}$ is trivial then the conditions (3) and (4) hold. Therefore we can immediately formulate the following corollary to Theorem 1:

Corollary 3. Consider N independent copies $(X_{ij})_{j \in \mathbb{Z}}$, $i = 1, \dots, N$ of a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ of real-valued random variables centered and in \mathbb{L}^2 with trivial left tail sigma field, $\mathcal{G}_{-\infty}$. Then the conclusion of Theorem 1 holds.

2 Some technical results for symmetric matrices

A key step in the proof of Theorem 1 is to show that the study of the limiting spectral distribution function of \mathbb{B}_N can be reduced to studying the same problem as for a Gaussian matrix with the same covariance structure. This step will be achieved with the help of some preliminary technical results concerning symmetric matrices with independent rows below the diagonal. These technical results have interest in themselves since they show that, for symmetric matrices with independent rows below the diagonal, very simple regularity conditions on the entries of each row allow to reduce the study of their limiting spectral distribution function to the one of a symmetric Gaussian matrix with the same covariance structure. In particular, this applies when the rows, below the diagonal, are independent and generated by the same stationary sequence provided it is regular, i.e. has a trivial left tail sigma-field.

To state the results of this section, let us introduce some notations. Let $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$ be real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, we consider the symmetric $n \times n$ random matrix \mathbf{X}_n defined as follows: for any i and j in $\{1, \dots, n\}$,

$$\begin{aligned} (\mathbf{X}_n)_{ij} &= X_{ij} \text{ for } i \geq j \text{ and} \\ (\mathbf{X}_n)_{ij} &= X_{ji} \text{ for } i < j. \end{aligned} \tag{6}$$

Define

$$\mathbb{X}_n := \frac{1}{n^{1/2}} \mathbf{X}_n, \tag{7}$$

and set

$$L(A) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^i \mathbb{E}(X_{ij}^2 I(|X_{ij}| > A)),$$

where A is a positive number.

We shall introduce now a Lindeberg's type condition:

Condition 4. (1) $\mathbb{E}(X_{\mathbf{u}}) = 0$ for all $\mathbf{u} \in \mathbb{N}^2$.
(2) There is $\sigma > 0$ such that $\sup_{\mathbf{u} \in \mathbb{N}^2} \|X_{\mathbf{u}}\|_2 \leq \sigma$.
(3) For every $\varepsilon > 0$ we have $L(\varepsilon n^{1/2}) \rightarrow 0$ as $n \rightarrow \infty$.

Clearly the items (2) and (3) of this condition are satisfied as soon as the family $(X_{\mathbf{u}}^2)$ is uniformly integrable or the random field is stationary.

Next result, in the nonstationary setting, shows that two mild regularity-like conditions without rates, are sufficient for reducing the study of the limiting spectral distribution of a symmetric matrix with independent rows below the diagonal to the corresponding problem for

a Gaussian matrix having the same covariance structure. This result indicates that for large matrix size, the empirical distribution of the eigenvalues is universal, in the sense that it is determined only by the covariance structure of the process.

Theorem 5. *Assume that Condition 4 is satisfied and in addition that the random vectors $(R_i)_{i \geq 1}$, where $R_i = (X_{ij})_{j \in \mathbb{N}}$, are mutually independent. For any $i \geq 1$ fixed, let $\mathcal{G}_{ik} = \sigma(X_{ij}, 1 \leq j \leq k)$ and, by convention, for $k \leq 0$, $\mathcal{G}_{ik} = \{\emptyset, \Omega\}$. Then, under the following two additional assumptions:*

$$\eta_m = \sup_{i \geq j \geq m} \|\mathbb{E}(X_{ij} | \mathcal{G}_{i,j-m})\|_2 \rightarrow 0 \quad (8)$$

and

$$\gamma_m = \sup_{i \geq \ell \geq k \geq m} \|\mathbb{E}(X_{ik} X_{i\ell} | \mathcal{G}_{i,k-m}) - \mathbb{E}(X_{ik} X_{i\ell})\|_1 \rightarrow 0, \quad (9)$$

the following convergence holds: for all $z \in \mathbb{C}^+$,

$$S^{\mathbb{X}_n}(z) - \mathbb{E}S^{\mathbb{Y}_n}(z) \rightarrow 0 \text{ almost surely, as } n \rightarrow \infty, \quad (10)$$

where \mathbb{X}_n is defined by (7) and $\mathbb{Y}_n = \mathbf{Y}_n / \sqrt{n}$, \mathbf{Y}_n being the symmetric matrix defined as in (6) and constructed from a centered real-valued Gaussian random field $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$ having the same covariance structure as $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$.

Remark 6. Since \mathbf{Y}_n is constructed from a centered real-valued Gaussian random field $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$ having the same covariance structure as $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$, we have in particular that the random vectors $(G_i)_{i \geq 1}$, where $G_i = (Y_{ij})_{j \in \mathbb{N}}$, are mutually independent. Therefore relation (16) in the proof of Theorem 5 also holds for \mathbb{Y}_n . Hence, in addition to the conclusion of Theorem 5, we also have

$$S^{\mathbb{X}_n}(z) - S^{\mathbb{Y}_n}(z) \rightarrow 0 \text{ almost surely, as } n \rightarrow \infty,$$

provided that $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$ and $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{N}^2}$ are defined on the same probability space.

Remark 7. Theorem 5 also holds if we allow the random variables X_{ij} to depend on the matrix size n . In this context we write $X_{ij}^{(n)}$ instead of X_{ij} , we adapt in a natural way Condition 1 and we modify conditions (8) and (9) as follows:

$$\sup_{n \geq 1} \sup_{i \geq j \geq m} \|\mathbb{E}(X_{ij}^{(n)} | \mathcal{G}_{i,j-m}^{(n)})\|_2 \xrightarrow{m \rightarrow \infty} 0$$

and

$$\sup_{n \geq 1} \sup_{i \geq \ell \geq k \geq m} \|\mathbb{E}(X_{ik}^{(n)} X_{i\ell}^{(n)} | \mathcal{G}_{i,k-m}^{(n)}) - \mathbb{E}(X_{ik}^{(n)} X_{i\ell}^{(n)})\|_1 \xrightarrow{m \rightarrow \infty} 0.$$

Next corollary applies to stationary sequences and shows that the conclusion of Theorem 5 holds under simple regularity conditions.

Corollary 8. *Let $(X_{ij})_{j \in \mathbb{Z}}$, $i = 1, \dots, n$ be n independent copies of a stationary sequence $(X_k)_{k \in \mathbb{Z}}$ of real-valued random variables which are centered and in \mathbb{L}^2 . Then the conclusion of Theorem 5 holds under the regularity conditions (3) and (4).*

Theorem 5 and its Remark 7 allow us to formulate the following result for Gram matrices. It will be a key step in the proof of Theorem 1.

Theorem 9. *Under the conditions of Theorem 5 and if $p/N \rightarrow c \in (0, \infty)$, the following convergence holds: for all $z \in \mathbb{C}^+$,*

$$S^{\mathbb{B}_N}(z) - \mathbb{E}S^{\mathbb{H}_N}(z) \rightarrow 0 \text{ almost surely, as } N \rightarrow \infty,$$

where \mathbb{B}_N is defined by (2) and \mathbb{H}_N is a Gram random matrix associated with a centered real-valued Gaussian process $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$ having the same covariance structure as $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$.

3 Examples

Below we give a few examples of regular processes.

1. Functions of i.i.d. random variables. Let $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}}$ be i.i.d. and $g : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be a measurable function such that, for any i, j in \mathbb{Z} , $X_{ij} = g(\varepsilon_{ik}, k \leq j)$ is well defined in \mathbb{L}^2 and $\mathbb{E}(X_{ij}) = 0$. These are regular random fields since each row has a trivial left sigma field. Therefore for these processes, conditions (3) and (4) are satisfied. Examples include linear processes, functions of linear processes and iterated random functions (see for instance Wu and Woodroffe (2000), among others).

For example let $X_{ij} = \sum_{k=0}^{\infty} a_k \varepsilon_{i, k-j}$, where ε_{ij} are i.i.d. with mean 0 and finite variance, and a_k are real coefficients with $\sum_{k=1}^{\infty} a_k^2 < \infty$. In this case X_{ij} is well-defined, the process is regular, and therefore the conclusion of Theorem 1 holds. The limiting empirical eigenvalue distribution of Gram matrices associated with linear processes was investigated in several papers (see for instance [21], [31] and [4]) but, all the previous known results treat only the short memory case meaning that the a_k 's are absolutely summable.

As we mentioned before, conditions (3) and (4) are satisfied for a stationary sequence if the left tail sigma field $\mathcal{G}_{-\infty}$ is trivial. Processes with trivial tail sigma field are called regular (see Chapter 2, Volume 1 in Bradley, 2007). We give next examples of regular processes.

1. Mixing sequences. The strong mixing coefficient is defined in the following way:

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\},$$

where \mathcal{A} and \mathcal{B} are two sigma algebras.

The ρ -mixing coefficient, also known as maximal coefficient of correlation, is defined as

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\{\text{Cov}(X, Y) / \|X\|_2 \|Y\|_2 : X \in \mathbb{L}^2(\mathcal{A}), Y \in \mathbb{L}^2(\mathcal{B})\}.$$

For the stationary sequence of random variables $(X_k)_{k \in \mathbb{Z}}$, \mathcal{F}^n denotes the σ -field generated by X_i with indices $i \geq n$, and \mathcal{F}_m denotes the σ -field generated by X_i with indices $i \leq m$. Then we define the sequences of mixing coefficients

$$\alpha_n = \alpha(\mathcal{F}_0, \mathcal{F}^n) \text{ and } \rho_n = \rho(\mathcal{F}_0, \mathcal{F}^n).$$

A sequence is called strongly mixing if $\alpha_n \rightarrow 0$. It is well-known that for strongly mixing sequences the left tail sigma field is trivial; see Claim 2.17a in Bradley (2007). Examples of this type include Harris recurrent Markov chains.

If $\lim_{n \rightarrow \infty} \rho_n < 1$, then the tail sigma field is also trivial according to Section 2.5 in Bradley (2005).

Note that our conditions (8) and (9) also hold without the assumptions of stationarity and of regularity. For instance, if

$$\alpha_{2,n} := \sup_{i \geq 1} \sup_{j \geq k} \alpha(\sigma(X_{i1}, \dots, X_{ik}), \sigma(X_{i, k+n}, X_{i, j+n})) \rightarrow 0,$$

and if the variables are centered and $(X_{\mathbf{u}}^2)_{\mathbf{u} \in \mathbb{Z}^2}$ is uniformly integrable, then (8) and (9) are satisfied. Note that the condition $\alpha_{2,n} \rightarrow 0$ is not enough for regularity.

For a nonstationary example we shall look at a more general linear process, based on martingale difference innovations satisfying Lindeberg's condition.

2. Linear processes with martingale entries. Assume that for any $1 \leq j \leq i \leq n$, the $(i, j)^{\text{th}}$ entry of \mathbf{X}_n is given by a linear process of the form

$$X_{ij} = \sum_{\ell=0}^{\infty} a_{i\ell} d_{i, j-\ell}, \tag{11}$$

where $(a_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$ is a sequence of real numbers and $(d_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$ is a sequence of real-valued random variables satisfying the conditions below:

A₁ $A_{n,i} = \sum_{j=0}^n a_{ij}^2 < \infty$ is convergent as $n \rightarrow \infty$ uniformly in $i \geq 1$.

A₂ There is $\sigma > 0$ such that $\sup_{\mathbf{u} \in \mathbb{Z}^2} \|d_{\mathbf{u}}\|_2 < \sigma$ and for every $\varepsilon > 0$,

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^i \mathbb{E}(d_{ij}^2 I(|d_{ij}| > \varepsilon \sqrt{n})) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

A₃ Setting $\mathcal{F}_{ij} = \sigma(d_{ik}, k \leq j)$, $\mathbb{E}(d_{ij} | \mathcal{F}_{i,j-1}) = 0$ a.s. for any (i, j) in \mathbb{Z}^2 and

$$\sup_{i \geq 1} \sup_{j \geq n} \|\mathbb{E}(d_{ij}^2 | \mathcal{F}_{i,j-n}) - \mathbb{E}(d_{ij}^2)\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Corollary 10. *Assume that (X_{ij}) is a linear process as defined in (11) such that the conditions **A₁**, **A₂** and **A₃** hold. Assume in addition that the random vectors $(d_i.)_{i \geq 1}$, where $d_i. = (d_{ij})_{j \in \mathbb{Z}}$, are mutually independent. Then the conclusion of Theorem 5 hold.*

The proof of this corollary is based on standard arguments for martingales and is left to the reader.

4 Proofs

4.1 Preparatory materials

In this section, we collect several results useful for our proofs.

The first result we mention is Lemma 2.1 in Götze *et al.* (2012) that allows to compare the difference between two Stieltjes transforms.

Lemma 11. *Let \mathbf{A} and \mathbf{B} be two symmetric $n \times n$ matrices with real entries. Then, for any $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$,*

$$|S_{\mathbf{A}}(z) - S_{\mathbf{B}}(z)| \leq \frac{1}{y^2 \sqrt{n}} |\text{Tr}(\mathbf{A} - \mathbf{B})|^{1/2}.$$

Relevant to the proof of Theorem 1 is the following lemma which gives an estimate of the Lévy distance between two distribution functions of eigenvalues (see Corollary A.42 in Bai and Silverstein (2010)).

Lemma 12. *Let \mathbf{A} and \mathbf{B} be two $n \times p$ matrices with real entries, and d be the Lévy distance. Then, for any $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$,*

$$d^2(F_{\mathbf{A}\mathbf{A}^T}, F_{\mathbf{B}\mathbf{B}^T}) \leq \frac{\sqrt{2}}{n} [\text{Tr}(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T) \text{Tr}((\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^T)]^{1/2}.$$

All along the proofs, we shall use the fact that the Stieltjes transform of the spectral measure is a smooth function of the matrix entries. To formalize things in a way that is suitable for our purpose, we shall adopt the same notations as in Chatterjee (2006) and introduce the following map A which "constructs" Wigner-type matrices. Let $N = n(n+1)/2$ and write elements of \mathbb{R}^N as $\mathbf{x} = (x_{ij})_{1 \leq j \leq i \leq n}$. For any \mathbf{x} in \mathbb{R}^N , let $A(\mathbf{x})$ be the matrix defined by

$$(A(\mathbf{x}))_{ij} = \begin{cases} \frac{1}{\sqrt{n}} x_{ij} & i \geq j \\ \frac{1}{\sqrt{n}} x_{ji} & i < j. \end{cases} \quad (12)$$

Let $z \in \mathbb{C}^+$ and $s_n := s_{n,z}$ be the function defined from \mathbb{R}^N to \mathbb{C} by

$$s_n(\mathbf{x}) = \frac{1}{n} \text{Tr}(A(\mathbf{x}) - z\mathbf{I}_n)^{-1}, \quad (13)$$

where \mathbf{I}_n is the identity matrix of order n .

The function s_n , as defined above, admits partial derivatives of all orders that are uniformly bounded. In particular, denoting for any $\mathbf{u} \in \{(i, j)\}_{1 \leq j \leq i \leq n}$, $\partial_{\mathbf{u}} s_n$ for $\partial s_n / \partial x_{\mathbf{u}}$, the following upper bounds hold: for any $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\{(i, j)\}_{1 \leq j \leq i \leq n}$, there exist universal positive constants c_1, c_2 and c_3 depending only on the imaginary part of z such that

$$|\partial_{\mathbf{u}} s_n| \leq \frac{c_1}{n^{3/2}}, \quad |\partial_{\mathbf{u}} \partial_{\mathbf{v}} s_n| \leq \frac{c_2}{n^2} \quad \text{and} \quad |\partial_{\mathbf{u}} \partial_{\mathbf{v}} \partial_{\mathbf{w}} s_n| \leq \frac{c_3}{n^{5/2}}. \quad (14)$$

(See Chatterjee (2006)). In addition, concerning the partial derivatives of second order, the following lemma will be also useful.

Lemma 13. *Let $z \in \mathbb{C}^+$ and $s_n := s_{n,z}$ be defined by (13). Let $(a_{ij})_{1 \leq j \leq i \leq n}$ and $(b_{ij})_{1 \leq j \leq i \leq n}$ be real numbers. Then, there exists an universal positive constant c_4 depending only on the imaginary part of z such that for any subset \mathcal{I}_n of $\{(i, j)\}_{1 \leq j \leq i \leq n}$ and any element \mathbf{x} of \mathbb{R}^N ,*

$$\left| \sum_{\mathbf{u} \in \mathcal{I}_n} \sum_{\mathbf{v} \in \mathcal{I}_n} a_{\mathbf{u}} b_{\mathbf{v}} \partial_{\mathbf{u}} \partial_{\mathbf{v}} s_n(\mathbf{x}) \right| \leq \frac{c_4}{n^2} \left(\sum_{\mathbf{u} \in \mathcal{I}_n} a_{\mathbf{u}}^2 \sum_{\mathbf{v} \in \mathcal{I}_n} b_{\mathbf{v}}^2 \right)^{1/2}.$$

Proof. Setting $G = (A(\mathbf{x}) - z\mathbf{I}_n)^{-1}$, we have

$$\partial_{\mathbf{u}} \partial_{\mathbf{v}} s_n = \frac{1}{n} \text{Tr}(G \partial_{\mathbf{u}} A G \partial_{\mathbf{v}} A G) + \frac{1}{n} \text{Tr}(G \partial_{\mathbf{v}} A G \partial_{\mathbf{u}} A G).$$

(See the equality (20) in Chatterjee (2006)). Whence, with the notations

$$\tilde{A} := \sum_{\mathbf{u} \in \mathcal{I}_n} a_{\mathbf{u}} \partial_{\mathbf{u}} A \quad \text{and} \quad \tilde{B} := \sum_{\mathbf{u} \in \mathcal{I}_n} b_{\mathbf{u}} \partial_{\mathbf{u}} A,$$

it follows that

$$\sum_{\mathbf{u} \in \mathcal{I}_n} \sum_{\mathbf{v} \in \mathcal{I}_n} a_{\mathbf{u}} b_{\mathbf{v}} \partial_{\mathbf{u}} \partial_{\mathbf{v}} s_n = \frac{2}{n} \text{Tr}(G^2 \tilde{A} G \tilde{B}).$$

Recall now the following facts: Let B and C be two complex valued matrices of order n . Then, $|\text{Tr}(BC)| \leq \|B\|_2 \|C\|_2$ where $\|B\|_2^2 = \sum_{i=1}^n \sum_{j=1}^n b_{ij}^2$ (the b_{ij} 's being the entries of B) and $\max\{\|BC\|_2, \|CB\|_2\} \leq \max_{1 \leq i \leq n} |\eta_i| \cdot \|C\|_2$ if B admits a spectral decomposition with eigenvalues η_1, \dots, η_n . Therefore using the above facts, together with the facts that $(\partial_{\mathbf{u}} A)_{ij} = n^{-1/2}$ if $(i, j) = \mathbf{u}$ or $(j, i) = \mathbf{u}$ and 0 otherwise, and that G admits a spectral decomposition with eigenvalues bounded by $1/y$ with $y = \text{Im}(z)$, we get

$$\frac{1}{n} |\text{Tr}(G^2 \tilde{A} G \tilde{B})| \leq \|G^2 \tilde{A}\|_2 \|G \tilde{B}\|_2 \leq \frac{1}{y^2} \frac{2}{n^2} \left(\sum_{\mathbf{u} \in \mathcal{I}_n} a_{\mathbf{u}}^2 \sum_{\mathbf{v} \in \mathcal{I}_n} b_{\mathbf{v}}^2 \right)^{1/2},$$

proving the lemma. \diamond

Another key result we use for dealing with Gaussian vectors is:

Lemma 14. *Let $X = (X_k)_{1 \leq k \leq n}$ and $Y = (Y_k)_{1 \leq k \leq n}$ be two vectors in \mathbb{L}^2 which have the same covariance structure. Assume in addition that Y is Gaussian. Then, for all $u \leq k$ we have*

$$\|\mathbb{E}(Y_k | \mathcal{F}_u^Y)\|_2 \leq \|\mathbb{E}(X_k | \mathcal{F}_u^X)\|_2,$$

where $\mathcal{F}_u^Y = \sigma(Y_i, i \leq u)$ and $\mathcal{F}_u^X = \sigma(X_i, i \leq u)$.

Proof. To prove the inequality above, it suffices to notice the following facts. Let

$$\mathcal{V}_u^Y = \overline{\text{span}}(1, (Y_j, 1 \leq j \leq u)) \text{ and } \mathcal{V}_u^X = \overline{\text{span}}(1, (X_j, 1 \leq j \leq u)),$$

where the closure is taken in \mathbb{L}^2 . Denote by $\Pi_{\mathcal{V}_u^Y}(\cdot)$ the orthogonal projection on \mathcal{V}_u^Y and by $\Pi_{\mathcal{V}_u^X}(\cdot)$ the orthogonal projection on \mathcal{V}_u^X . Since $(Y_j)_{1 \leq j \leq n}$ is a Gaussian vector $\mathbb{E}(Y_k | \mathcal{F}_u^Y) = \Pi_{\mathcal{V}_u^Y}(Y_k)$ a.s. and in \mathbb{L}^2 . On another hand, since $(Y_k)_{1 \leq k \leq n}$ has the same covariance structure as $(X_k)_{1 \leq k \leq n}$, we observe that

$$\|\Pi_{\mathcal{V}_u^Y}(Y_k)\|_2 = \|\Pi_{\mathcal{V}_u^X}(X_k)\|_2.$$

But, by the definition of the conditional expectation, $\|X_k - \mathbb{E}(X_k | \mathcal{F}_u^X)\|_2 \leq \|X_k - \Pi_{\mathcal{V}_u^X}(X_k)\|_2$. Hence, by Pythagora's theorem,

$$\|\Pi_{\mathcal{V}_u^X}(X_k)\|_2 \leq \|\mathbb{E}(X_k | \mathcal{F}_u^X)\|_2.$$

Combining all the observations above, the lemma follows. \diamond

Our next proposition gives in particular a well-known linear representation for stationary Gaussian processes which have a spectral density. It can be found in Varadhan (Ch 6, Section 6.6., (2001)); see also Fact 3.1 in Chakrabarty *et al.* (2014).

Proposition 15. *Let f be the spectral density on $[-\pi, \pi]$ of a real-valued \mathbb{L}^2 -stationary process. For any $k \in \mathbb{Z}$, let*

$$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ikx} \sqrt{f(x)} dx. \quad (15)$$

Then the a_k 's are real numbers satisfying $\sum_{k \in \mathbb{Z}} a_k^2 < \infty$. In addition, if we define for any $k \in \mathbb{Z}$,

$$Y_k = \sum_{j \in \mathbb{Z}} a_j \xi_{k-j},$$

where $(\xi_j)_{j \in \mathbb{Z}}$ is a sequence of i.i.d. standard normal real-valued random variables, then $(Y_k)_{k \in \mathbb{Z}}$ is a centered real-valued stationary Gaussian process with spectral density f on $[-\pi, \pi]$.

4.2 Proof of Theorem 5

The proof of this theorem requires several steps. First we reduce the problem to studying the difference of expected values. Next, in order to weaken the dependence, we partition the variables in each row in big and small blocks. The big blocks are approximated by vector valued martingale differences. Then, we replace one by one these martingale differences by Gaussian random vectors having the same covariance structure with the help of a blockwise Lindeberg-type method.

All along the proof $z = x + iy$ will be a complex number in \mathbb{C}^+ . Also, the notation $a \ll b$ means that there is a constant C depending only on $\text{Im } z = y$ such that $a \leq Cb$.

Step 1: Reduction of the problem to a difference of expected values.

Since the random vectors $(R_i)_{1 \leq i \leq n}$, where $R_i = (X_{ij})_{1 \leq j \leq i}$, are mutually independent, it is well-known (see for instance the arguments in the proof on page 34 in Bai-Silverstein, 2010) that

$$S^{\mathbb{X}_n}(z) - \mathbb{E}S^{\mathbb{X}_n}(z) \rightarrow 0 \text{ a.s.} \quad (16)$$

Hence, in order to prove Theorem 5, it suffices to show that

$$\mathbb{E}S^{\mathbb{X}_n}(z) - \mathbb{E}S^{\mathbb{Y}_n}(z) \rightarrow 0. \quad (17)$$

To prove the above convergence, notice that there is no loss of generality in assuming that the entries $(Y_{\mathbf{u}})$ of \mathbf{Y}_n are independent of the entries $(X_{\mathbf{u}})$ of \mathbf{X}_n . Therefore, from now on, we assume that \mathbf{Y}_n is a symmetric matrix constructed from a real-valued centered Gaussian random field $(Y_{\mathbf{u}})$ having the same covariance structure as $(X_{\mathbf{u}})$ and independent of $(X_{\mathbf{u}})$.

We write $S^{\mathbb{X}_n}(z)$ and $S^{\mathbb{Y}_n}(z)$ as a function of the entries on and below the diagonal, arranged row after row. More exactly, using the notation (13), we write

$$S^{\mathbb{X}_n}(z) = s_n(L^X) \text{ and } S^{\mathbb{Y}_n}(z) = s_n(L^Y),$$

where $L^X = (L_i^X)_{1 \leq i \leq n}$ and $L^Y = (L_i^Y)_{1 \leq i \leq n}$ with $L_i^X = (X_{i1}, \dots, X_{ii})$ and $L_i^Y = (Y_{i1}, \dots, Y_{ii})$. Also, in the sequel, to further simplify the notation we shall skip the index n from s_n and we put $s = s_n := s_{n,z}$.

Step 2: Martingale approximation.

We shall introduce a martingale structure on each row. We start from the celebrated Bernstein big and small blocks argument which weakens the dependence. We partition the variables in each row in big and small blocks and show that the variables in large blocks have a dominant contribution. The large blocks are then decomposed in martingale differences and a rest which also has a smaller contribution.

Let p and q be two integers fixed for the moment. Fix i in $\{1, \dots, n\}$ and let $k_i = \lfloor i/(p+q) \rfloor$. We partition the set $\{1, \dots, i\}$ in big and small blocks with the following restriction: a big block of size p is followed by a small block of size q . We shall have the set of indexes $I_{i1}, J_{i1}, I_{i2}, J_{i2}, \dots, I_{ik_i}, J_{ik_i}, J_{i,k_i+1}$ where each index set I_{ij} is of size p , each index set J_{ij} is of size q and the last block has a size at most $p+q$. More precisely, for any i in $\{1, \dots, n\}$ and for any $j \in \{1, \dots, k_i\}$,

$$I'_j = \{(j-1)(p+q) + 1, \dots, (j-1)(p+q) + p\} \text{ and } J'_j = \{(j-1)(p+q) + p + 1, \dots, j(p+q)\}.$$

and

$$I_{ij} = \{(i, k); k \in I'_j\} \text{ and } J_{ij} = \{(i, k); k \in J'_j\}.$$

Corresponding to this index decomposition, the vectors L_j^X and L_j^Y are partitioned in $k_i + 1$ consecutive vectors. Setting

$$B_{ij} = (X_{\mathbf{u}})_{\mathbf{u} \in I_{ij}}, b_{ij} = (X_{\mathbf{u}})_{\mathbf{u} \in J_{ij}}, B_{ij}^* = (Y_{\mathbf{u}})_{\mathbf{u} \in I_{ij}} \text{ and } b_{ij}^* = (Y_{\mathbf{u}})_{\mathbf{u} \in J_{ij}}$$

we write

$$L_i^X = (B_{i1}, b_{i1}, B_{i2}, b_{i2}, \dots, B_{ik_i}, b_{ik_i}, b_{i,k_i+1}) \text{ and } L_i^Y = (B_{i1}^*, b_{i1}^*, B_{i2}^*, b_{i2}^*, \dots, B_{ik_i}^*, b_{ik_i}^*, b_{i,k_i+1}^*).$$

We introduce now the following vectors

$$B_i^X = (B_{i1}, \mathbf{0}_q, B_{i2}, \mathbf{0}_q, \dots, B_{ik_i}, \mathbf{0}_q, \mathbf{0}_r) \text{ and } B_i^Y = (B_{i1}^*, \mathbf{0}_q, B_{i2}^*, \mathbf{0}_q, \dots, B_{ik_i}^*, \mathbf{0}_q, \mathbf{0}_r),$$

where $r = i - k_i(p+q)$. Note that B_i^X (resp. B_i^Y) is derived from L_i^X (resp. L_i^Y) where we replace the variables in b_{ij} (resp. b_{ij}^*) by 0's. In addition, for A a positive real, fixed for the moment, we set for any $\mathbf{u} \in \mathbb{Z}^2$

$$\tilde{X}_{\mathbf{u}} := X_{\mathbf{u}} I(|X_{\mathbf{u}}| \leq A),$$

and, for any $i \in \{1, \dots, n\}$,

$$\tilde{B}_i^X = (\tilde{B}_{i1}, \mathbf{0}_q, \tilde{B}_{i2}, \mathbf{0}_q, \dots, \tilde{B}_{ik_i}, \mathbf{0}_q, \mathbf{0}_r) \text{ where } \tilde{B}_{ij} = (\tilde{X}_{\mathbf{u}})_{\mathbf{u} \in I_{ij}} \text{ for } j \in \{1, \dots, k_i\}.$$

Next, for any $i \in \{1, \dots, n\}$, we consider the sigma algebras $\mathcal{F}_{i0}^X = \mathcal{F}_{i0}^Y = \{\emptyset, \Omega\}$ and for $1 \leq \ell \leq k_i$, $\mathcal{F}_{i\ell}^X = \sigma(B_{ij}; 1 \leq j \leq \ell)$ and $\mathcal{F}_{i\ell}^Y = \sigma(B_{ij}^*; 1 \leq j \leq \ell)$. Then, for any $\ell \in \{1, \dots, k_i\}$, we define

$$\tilde{D}_{i\ell} = \tilde{B}_{i\ell} - \mathbb{E}(\tilde{B}_{i\ell} | \mathcal{F}_{i,\ell-1}^X), \quad (18)$$

and

$$D_{i\ell}^* = B_{i\ell}^* - \mathbb{E}(B_{i\ell}^* | \mathcal{F}_{i,\ell-1}^Y). \quad (19)$$

By $\mathbb{E}(\tilde{B}_{i\ell} | \mathcal{F}_{i,\ell-1}^X)$ (resp. $\mathbb{E}(B_{i\ell}^* | \mathcal{F}_{i,\ell-1}^Y)$) we understand a vector of dimension p where each component is a component of the vector $\tilde{B}_{i\ell}$ (resp. $B_{i\ell}^*$) conditioned with respect to $\mathcal{F}_{i,\ell-1}^X$ (resp. $\mathcal{F}_{i,\ell-1}^Y$). Note that $(\tilde{D}_{i\ell})_{1 \leq \ell \leq k_i}$ and $(D_{i\ell}^*)_{1 \leq \ell \leq k_i}$ are vector valued martingale differences adapted respectively to $(\mathcal{F}_{i\ell}^X)_{1 \leq \ell \leq k_i}$ and $(\mathcal{F}_{i\ell}^Y)_{1 \leq \ell \leq k_i}$. We then define the vectors \tilde{D}_i^X and D_i^Y with dimension i and with a similar structure as B_i^X as follows:

$$\tilde{D}_i^X = (\tilde{D}_{i1}, \mathbf{0}_q, \tilde{D}_{i2}, \mathbf{0}_q, \dots, \tilde{D}_{ik_i}, \mathbf{0}_q, \mathbf{0}_r) \text{ and } D_i^Y = (D_{i1}^*, \mathbf{0}_q, D_{i2}^*, \mathbf{0}_q, \dots, D_{ik_i}^*, \mathbf{0}_q, \mathbf{0}_r). \quad (20)$$

Setting $\tilde{D}^X = (\tilde{D}_i^X)_{1 \leq i \leq n}$, we first compare $\mathbb{E}s(L^X)$ to $\mathbb{E}s(\tilde{D}^X)$. We write

$$\mathbb{E}s(L^X) - \mathbb{E}s(\tilde{D}^X) = \mathbb{E}\Delta_1(s) + \mathbb{E}\Delta_2(s) + \mathbb{E}\Delta_3(s),$$

where

$$\Delta_1(s) = s(L^X) - s(B^X), \Delta_2(s) = s(B^X) - s(\tilde{B}^X)$$

and

$$\Delta_3(s) = s(\tilde{B}^X) - s(\tilde{D}^X),$$

with the notations $B^X = (B_i^X)_{1 \leq i \leq n}$ and $\tilde{B}^X = (\tilde{B}_i^X)_{1 \leq i \leq n}$. To control each of the $\mathbb{E}\Delta_i(s)$ for $i = 1, 2, 3$, we apply Lemma 11. Therefore, we get

$$|\mathbb{E}\Delta_1(s)|^2 \ll \sum_{i=1}^n \sum_{j=1}^{k_i+1} \sum_{\mathbf{u} \in J_{ij}} \mathbb{E}(X_{\mathbf{u}}^2) \ll \left(\frac{q}{p} + \frac{q+p}{n}\right) \sigma^2,$$

$$|\mathbb{E}\Delta_2(s)|^2 \ll \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^{k_i} \sum_{\mathbf{u} \in I_{ij}} \mathbb{E}(X_{\mathbf{u}}^2 I(|X_{\mathbf{u}}| > A)) \ll L(A),$$

and

$$\begin{aligned} |\mathbb{E}\Delta_3(s)|^2 &\ll \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^{k_i} \sum_{\mathbf{u} \in I_{ij}} \|\mathbb{E}(\tilde{X}_{\mathbf{u}} | \mathcal{F}_{i,j-1}^X)\|_2^2 \leq 2 \left(L(A) + \max_{1 \leq j \leq i \leq n} \|\mathbb{E}(X_{ij} | \mathcal{G}_{i,j-q})\|_2^2 \right) \\ &\ll (L(A) + \eta_q^2). \end{aligned}$$

We proceed in a similar way with the matrix \mathbb{Y}_n . Therefore, setting $D^Y = (D_i^Y)_{1 \leq i \leq n}$, we write

$$\mathbb{E}s(L^Y) - \mathbb{E}s(D^Y) = \mathbb{E}\Delta'_1(s) + \mathbb{E}\Delta'_2(s),$$

with the notations

$$\Delta'_1(s) = s(L^Y) - s(B^Y) \text{ and } \Delta'_2(s) = s(B^Y) - s(D^Y),$$

where $B^Y = (B_i^Y)_{1 \leq i \leq n}$. Applying Lemma 11 and using the fact that $(Y_{\mathbf{u}})$ has the same covariance structure as $(X_{\mathbf{u}})$, we derive

$$|\mathbb{E}\Delta'_1(s_n)|^2 \ll \left(\frac{q}{p} + \frac{q+p}{n}\right) \sup \mathbb{E}(Y_{\mathbf{u}}^2) \ll \left(\frac{q}{p} + \frac{q+p}{n}\right) \sigma^2.$$

On another hand, Lemmas 11 and 14 imply that

$$\begin{aligned} |\mathbb{E}\Delta'_2(s)|^2 &\ll \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^{k_i} \sum_{\mathbf{u} \in I_{ij}} \|\mathbb{E}(Y_{\mathbf{u}} | \mathcal{F}_{i,j-1}^Y)\|_2^2 \ll \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^{k_i} \sum_{\mathbf{u} \in I_{ij}} \|\mathbb{E}(X_{\mathbf{u}} | \mathcal{F}_{i,j-1}^X)\|_2^2 \\ &\ll \max_{1 \leq j \leq i \leq n} \|\mathbb{E}(X_{ij} | \mathcal{G}_{i,j-q})\|_2^2 \ll \eta_q^2. \end{aligned}$$

Overall we have the decomposition

$$\mathbb{E}S^{\mathbb{X}_n}(z) - \mathbb{E}S^{\mathbb{Y}_n}(z) = \mathbb{E}s(\tilde{D}^X) - \mathbb{E}s(D^Y) + E_n(p, q, A), \quad (21)$$

with

$$|E_n(p, q, A)| \ll \left(\left(\frac{q}{p} + \frac{q+p}{n} \right)^{1/2} \sigma + L^{1/2}(A) + \eta_q \right).$$

Step 3: The study of $\mathbb{E}s(\tilde{D}^X) - \mathbb{E}s(D^Y)$.

To study $\mathbb{E}s(\tilde{D}^X) - \mathbb{E}s(D^Y)$ we first decompose the difference according to the rows and after that we study the rows separately. With this aim we introduce a telescoping sum where each term is a difference of two functions whose arguments differ only by one row. Namely we write

$$\mathbb{E}s(\tilde{D}^X) - \mathbb{E}s(D^Y) = \sum_{i=1}^n \left(\mathbb{E}s(\tilde{D}_{[1,i-1]}^X, \tilde{D}_i^X, D_{[i+1,n]}^Y) - \mathbb{E}s(\tilde{D}_{[1,i-1]}^X, D_i^Y, D_{[i+1,n]}^Y) \right)$$

where $\tilde{D}_{[a,b]}^X = (\tilde{D}_a^X, \dots, \tilde{D}_b^X)$ and $D_{[a,b]}^Y = (D_a^Y, \dots, D_b^Y)$ with \tilde{D}_i^X and D_i^Y defined in (20). Now for every i fixed denote by

$$s_i(\mathbf{x}) := s(\tilde{D}_{[1,i-1]}^X, \mathbf{x}, D_{[i+1,n]}^Y).$$

Note that s_i is a random function from \mathbb{R}^i to \mathbb{C} . With this notation

$$\mathbb{E}s(\tilde{D}^X) - \mathbb{E}s(D^Y) = \sum_{i=1}^n \mathbb{E}(s_i(\tilde{D}_i^X) - s_i(D_i^Y)).$$

From now on, for easier notation, it will be convenient to extend the vectors $(\tilde{D}_{i\ell})_{1 \leq \ell \leq k_i}$ and $(D_{i\ell}^*)_{1 \leq \ell \leq k_i}$ defined in (18) and (19) as follows:

$$\tilde{D}'_{i\ell} = (\tilde{D}_{i\ell}, \mathbf{0}_q) \text{ and } D'_{i\ell}^* = (\tilde{D}_{i\ell}^*, \mathbf{0}_q) \text{ for } 1 \leq \ell \leq k_i - 1 \quad (22)$$

and

$$\tilde{D}'_{ik_i} = (\tilde{D}_{ik_i}, \mathbf{0}_{q+r}) \text{ and } D'_{ik_i}^* = (D_{ik_i}^*, \mathbf{0}_{q+r}). \quad (23)$$

With these notations, as in the Lindeberg's method, we write now another telescoping sum where we change one by one the vectors $\tilde{D}'_{i\ell}$ by $D'_{i\ell}^*$ in the argument of s_i . With this aim we write

$$\begin{aligned} s_i(\tilde{D}_i^X) - s_i(D_i^Y) &= s_i(\tilde{D}'_{i1}, \dots, \tilde{D}'_{ik_i}) - s_i(D'_{i1}^*, \dots, D'_{ik_i}^*) \\ &= \sum_{u=1}^{k_i} (s_i(\tilde{D}'_{i,[1,u-1]}, \tilde{D}'_{iu}, D'_{i,[u+1,k_i]}^*) - s_i(\tilde{D}'_{i,[1,u-1]}, D'_{iu}^*, D'_{i,[u+1,k_i]}^*)) \\ &:= \sum_{u=1}^{k_i} (s_{i,u}(\tilde{D}'_{iu}) - s_{i,u}(D'_{iu}^*)), \end{aligned} \quad (24)$$

where $\tilde{D}'_{i,[k,\ell]} := (\tilde{D}'_{ik}, \dots, \tilde{D}'_{i\ell})$ and $D'_{i,[k,\ell]}^* := (D'_{ik}^*, \dots, D'_{i\ell}^*)$. Note that the s_{iu} 's defined above are random functions from \mathbb{R}^{p+q} to \mathbb{C} if $1 \leq u \leq k_i - 1$ and from \mathbb{R}^{q+r} to \mathbb{C} if $u = k_i$ (where $r = i - k_i(p + q)$).

We shall treat separately each term in the sum (24) corresponding to the i -th row. So, in the following, i is fixed. To facilitate the study of this difference we introduce some auxiliary terms:

$$s_{iu}(\tilde{D}'_{iu}) - s_{iu}(D'_{iu}^*) = s_{iu}(\tilde{D}'_{iu}) - s_{iu}(\mathbf{0}) + s_{iu}(\mathbf{0}) - s_{iu}(D'_{iu}^*).$$

Denote by $d_{iu}^{(j)}$ the j -th component of the vector \tilde{D}'_{iu} . Using Taylor's expansion of order three, we get

$$s_{iu}(\tilde{D}'_{iu}) - s_{iu}(\mathbf{0}) = \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3, \quad (25)$$

where

$$\tilde{R}_1 = \sum_{j=1}^p d_{iu}^{(j)} \partial_j s_{iu}(\mathbf{0}), \quad \tilde{R}_2 = \frac{1}{2} \left(\sum_{j=1}^p d_{iu}^{(j)} \partial_j \right)^2 s_{iu}(\mathbf{0})$$

and

$$\tilde{R}_3 = \frac{1}{6} \left(\sum_{j=1}^p d_{iu}^{(j)} \partial_j \right)^3 s_{iu}(\theta \tilde{D}'_{iu}) \text{ with } \theta \in]0, 1[.$$

Similarly, if we denote by $g_{iu}^{(j)}$ the j -th component of the vector D'^*_{iu} , we get

$$s_{iu}(D'^*_{iu}) - s_{iu}(\mathbf{0}) = R_1^* + R_2^* + R_3^*, \quad (26)$$

where

$$R_1^* = \sum_{j=1}^p g_{iu}^{(j)} \partial_j s_{iu}(\mathbf{0}) \text{ and } R_2^* = \frac{1}{2} \left(\sum_{j=1}^p g_{iu}^{(j)} \partial_j \right)^2 s_{iu}(\mathbf{0})$$

and

$$R_3^* = \frac{1}{6} \left(\sum_{j=1}^p g_{iu}^{(j)} \partial_j \right)^3 s_{iu}(\theta D'^*_{iu}) \text{ with } \theta \in]0, 1[.$$

Now notice that, for any $u \in \{1, \dots, k_i\}$ and any $j \in \{1, \dots, p\}$,

$$d_{iu}^{(j)} = \tilde{X}_{i,(u-1)(p+q)+j} - \mathbb{E}(\tilde{X}_{i,(u-1)(p+q)+j} | \mathcal{F}_{i,u-1}^X) := \tilde{X}_{iu}^{(j)} - \mathbb{E}(\tilde{X}_{iu}^{(j)} | \mathcal{F}_{i,u-1}^X), \quad (27)$$

and

$$g_{iu}^{(j)} = Y_{i,(u-1)(p+q)+j} - \mathbb{E}(Y_{i,(u-1)(p+q)+j} | \mathcal{F}_{i,u-1}^Y) := Y_{iu}^{(j)} - \mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y). \quad (28)$$

Therefore

$$\|d_{iu}^{(j)}\|_3^3 \leq 2^3 \|\tilde{X}_{iu}^{(j)}\|_3^3 \ll A\sigma^2,$$

and since \mathbf{Y}_n has the same covariance structure as \mathbf{X}_n and is a Gaussian vector,

$$\|g_{iu}^{(j)}\|_3^3 \leq 2^3 \|Y_{iu}^{(j)}\|_3^3 \leq 2^4 \|Y_{iu}^{(j)}\|_2^3 \ll \sigma^3.$$

Taking into account the two previous inequalities and the upper bound on the partial derivatives of order three of s given in (14), we infer that

$$|\mathbb{E}(\tilde{R}_3) + \mathbb{E}(R_3^*)| \ll \frac{1}{n^{5/2}} p^3 \sigma^2 (A + \sigma). \quad (29)$$

On another hand, we notice that for any j, ℓ in $\{1, \dots, p\}$, $\partial_j s_{iu}(\mathbf{0})$ and $\partial_j \partial_\ell s_{iu}(\mathbf{0})$ are complex-valued random variables measurable with respect to the sigma algebra $\mathcal{H}_{i,u}$ defined by

$$\mathcal{H}_{i,u} = \mathcal{F}_{i,u-1}^X \vee \sigma((L_j^X)_{1 \leq j \leq i-1}, (L_k^Y)_{i+1 \leq k \leq n}) \vee \sigma(D_{i,u+1}^*, \dots, D_{ik_i}^*). \quad (30)$$

Hence

$$\mathbb{E}(\tilde{R}_1) = \sum_{j=1}^p \mathbb{E}(\partial_j s_{iu}(\mathbf{0}) \mathbb{E}(d_{iu}^{(j)} | \mathcal{H}_{i,u})),$$

and

$$\mathbb{E}(\tilde{R}_2) = \frac{1}{2} \sum_{j,\ell=1}^p \mathbb{E}(\partial_j \partial_\ell s_{iu}(\mathbf{0}) \mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{H}_{i,u})).$$

Since the rows of \mathbf{X}_n are assumed to be independent and \mathbf{Y}_n is assumed to be independent of \mathbf{X}_n , then $\sigma(d_{iu}^{(1)}, \dots, d_{iu}^{(p)}) \vee \mathcal{F}_{i,u-1}^X$ is independent of

$$\sigma((L_j^X)_{1 \leq j \leq i-1}, (L_k^Y)_{i+1 \leq k \leq n}) \vee \sigma(D_{i,u+1}^*, \dots, D_{ik_i}^*).$$

Therefore, by the properties of the conditional expectation, $\mathbb{E}(d_{iu}^{(j)} | \mathcal{H}_{i,u}) = \mathbb{E}(d_{iu}^{(j)} | \mathcal{F}_{i,u-1}^X) = 0$ and $\mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{H}_{i,u}) = \mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X)$. Hence,

$$\mathbb{E}(\tilde{R}_1) = 0 \text{ and } \mathbb{E}(\tilde{R}_2) = \frac{1}{2} \sum_{j,\ell=1}^p \mathbb{E}(\mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) \partial_j \partial_\ell s_{iu}(\mathbf{0})). \quad (31)$$

We handle now the terms $\mathbb{E}(R_1^*)$ and $\mathbb{E}(R_2^*)$. With this aim we notice that by definition $(D_{iu}^* : 1 \leq u \leq k_i)_{1 \leq i \leq n}$ is a centered Gaussian vector such that $\text{Cov}(D_{iu}^*, D_{i'u'}^*) = \mathbf{0}_{p,p}$ if $(i, u) \neq (i', u')$. Therefore $D_{i,u}^*, i = 1, \dots, n, u = 1, \dots, k_i$ are centered Gaussian random variables in \mathbb{R}^p which are mutually independent. In addition they are independent of $(X_{\mathbf{u}})$. Therefore,

$$\mathbb{E}(R_1^*) = \sum_{j=1}^p \mathbb{E}(g_{iu}^{(j)}) \mathbb{E}(\partial_j s_{iu}(\mathbf{0})) = 0, \quad (32)$$

and

$$\mathbb{E}(R_2^*) = \frac{1}{2} \sum_{j,\ell=1}^p \mathbb{E}(g_{iu}^{(j)} g_{iu}^{(\ell)}) \mathbb{E}(\partial_j \partial_\ell s_{iu}(\mathbf{0})). \quad (33)$$

So, starting from (24) and taking into account (25), (26), (29), (31), (32) and (33), we derive that for any $i \in \{1, \dots, n\}$,

$$\begin{aligned} \mathbb{E}(s_i(\tilde{D}_i^X)) - \mathbb{E}(s_i(D_i^Y)) &\ll \left| \sum_{u=1}^{k_i} \sum_{j,\ell=1}^p \mathbb{E} \left((\mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(g_{iu}^{(j)} g_{iu}^{(\ell)})) \partial_j \partial_\ell s_{iu}(\mathbf{0}) \right) \right| \\ &\quad + \frac{1}{n^{5/2}} k_i p^3 \sigma^2 (A + \sigma). \end{aligned} \quad (34)$$

We handle now the first term in the right-hand side of the above inequality. Recalling the notations (27) and (28), we first write

$$\begin{aligned} \mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(g_{iu}^{(j)} g_{iu}^{(\ell)}) &= \mathbb{E}(\tilde{X}_{iu}^{(j)} \tilde{X}_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(Y_{iu}^{(j)} Y_{iu}^{(\ell)}) \\ &\quad - \mathbb{E}(\tilde{X}_{iu}^{(j)} | \mathcal{F}_{i,u-1}^X) \mathbb{E}(\tilde{X}_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) + \mathbb{E}(\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y) \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y)). \end{aligned}$$

Therefore, by triangle inequality and Jensen inequality,

$$\begin{aligned} &\left| \sum_{u=1}^{k_i} \sum_{j,\ell=1}^p \mathbb{E} \left((\mathbb{E}(d_{iu}^{(j)} d_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(g_{iu}^{(j)} g_{iu}^{(\ell)})) \partial_j \partial_\ell s_{iu}(\mathbf{0}) \right) \right| \\ &\leq \sum_{u=1}^{k_i} \sum_{j,\ell=1}^p \left| \mathbb{E} \left((\mathbb{E}(\tilde{X}_{iu}^{(j)} \tilde{X}_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(Y_{iu}^{(j)} Y_{iu}^{(\ell)})) \partial_j \partial_\ell s_{iu}(\mathbf{0}) \right) \right| \\ &\quad + \sum_{u=1}^{k_i} \mathbb{E} \left| \sum_{j,\ell=1}^p \mathbb{E}(\tilde{X}_{iu}^{(j)} | \mathcal{F}_{i,u-1}^X) \mathbb{E}(\tilde{X}_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^X) \partial_j \partial_\ell s_{iu}(\mathbf{0}) \right| \\ &\quad + \sum_{u=1}^{k_i} \left| \sum_{j,\ell=1}^p \mathbb{E}(\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y) \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y)) \partial_j \partial_\ell s_{iu}(\mathbf{0}) \right| \\ &:= T_1 + T_2 + T_3. \end{aligned} \quad (35)$$

Let us first handle T_3 . Recalling the notation (22) and (23) and setting

$$C_{i,u} = (\tilde{D}_{[1,i-1]}^X, \tilde{D}'_1, \dots, \tilde{D}'_{i,u-1}, \mathbf{0}, D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}, D^Y_{[i+1,n]}), \quad (36)$$

we note that $\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y) \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y)$ is independent of $\partial_j \partial_\ell s(C_{i,u})$. This is because of the independence between \mathbf{Y}_n and \mathbf{X}_n together with the independence between the vectors $(\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y), \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y))$ and $(D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}, D^Y_{[i+1,n]})$. To prove the latter independence, it suffices to notice that $(\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y), \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y), D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}, D^Y_{[i+1,n]})$ is a Gaussian vector and that $(\mathbb{E}(Y_{iu}^{(j)} | \mathcal{F}_{i,u-1}^Y), \mathbb{E}(Y_{iu}^{(\ell)} | \mathcal{F}_{i,u-1}^Y))$ and $(D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}, D^Y_{[i+1,n]})$ are uncorrelated. So, we can bound T_3 as follows:

$$T_3 \leq \sum_{u=1}^{k_i} \mathbb{E} \left| \sum_{j,k \in I'_u} \mathbb{E}(Y_{ij} | \mathcal{F}_{i,u-1}^Y) \mathbb{E}(Y_{ik} | \mathcal{F}_{i,u-1}^Y) \partial_{ij} \partial_{ik} s(C_{i,u}) \right|.$$

An application of Lemma 13 gives

$$\left| \sum_{j,k \in I'_u} \mathbb{E}(Y_{ij} | \mathcal{F}_{i,u-1}^Y) \mathbb{E}(Y_{ik} | \mathcal{F}_{i,u-1}^Y) \partial_{ij} \partial_{ik} s(C_{i,u}) \right| \ll \frac{1}{n^2} \sum_{j \in I'_u} (\mathbb{E}(Y_{ij} | \mathcal{F}_{i,u-1}^Y))^2.$$

Whence, using in addition Lemma 14, we derive

$$T_3 \ll \frac{1}{n^2} \sum_{u=1}^{k_i} \sum_{j \in I'_u} \|\mathbb{E}(Y_{ij} | \mathcal{F}_{i,u-1}^Y)\|_2^2 \ll \frac{1}{n^2} \sum_{u=1}^{k_i} \sum_{j \in I'_u} \|\mathbb{E}(X_{ij} | \mathcal{F}_{i,u-1}^X)\|_2^2.$$

Since $\mathcal{F}_{i,u-1}^X \subset \mathcal{G}_{i,\ell-q}$ for any $\ell \in \{(u-1)(p+q)+1, \dots, (u-1)(p+q)+p\}$, it follows that

$$T_3 \ll \frac{1}{n^2} \sum_{j=1}^i \|\mathbb{E}(X_{ij} | \mathcal{G}_{i,j-q})\|_2^2 \ll \frac{1}{n} \eta_q^2. \quad (37)$$

To treat T_2 we proceed as in the proof of relation (37), and infer that

$$T_2 \ll \frac{1}{n^2} \sum_{j=1}^i \|\mathbb{E}(\tilde{X}_{ij} | \mathcal{G}_{i,j-q})\|_2^2 \ll \frac{1}{n} \eta_q^2 + \frac{1}{n^2} \sum_{j=1}^i \|X_{ij}^2 I(|X_{ij}| > A)\|_1. \quad (38)$$

We handle now the term T_1 in (35). Using the notation (36) and the fact that \mathbf{Y}_n has the same covariance structure as \mathbf{X}_n , we start by rewriting T_1 as follows:

$$\begin{aligned} T_1 &= \sum_{u=1}^{k_i} \sum_{j, \ell \in I'_u} |\mathbb{E}((\mathbb{E}(\tilde{X}_{ij} \tilde{X}_{i\ell} | \mathcal{F}_{i,u-1}^X) - \mathbb{E}(X_{ij} X_{i\ell})) \partial_{ij} \partial_{i\ell} s(C_{i,u}))| \\ &= \sum_{u=1}^{k_i} \sum_{j, \ell \in I'_u} |\mathbb{E}((\tilde{X}_{ij} \tilde{X}_{i\ell} - \mathbb{E}(X_{ij} X_{i\ell})) \partial_{ij} \partial_{i\ell} s(C_{i,u}))|, \end{aligned} \quad (39)$$

where for the second equality we used the fact that $\partial_{ij} \partial_{i\ell} s(C_{i,u})$ is measurable with respect to $\mathcal{H}_{i,u}$ defined by (30) and that $\sigma((X_{i,(u-1)(p+q)+j})_{1 \leq j \leq p}) \vee \mathcal{F}_{i,u-1}^X$ is independent of

$$\sigma((L_j^X)_{1 \leq j \leq i-1}, (L_k^Y)_{i+1 \leq k \leq n}) \vee \sigma(D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}).$$

To treat the summands in (39), we further weaken the dependence by suppressing some variables in $C_{i,u}$ which are "close" to $\tilde{X}_{ij} \tilde{X}_{i\ell}$. Let a be a positive integer fixed for the moment. Then, setting,

$$C_{i,u}^{(a)} = (\tilde{D}_{[1,i-1]}^X, \tilde{D}'_1, \dots, \tilde{D}'_{i,u-a}, \mathbf{0}, D'^*_{i,u+1}, \dots, D'^*_{i,u_{k_i}}, D^Y_{[i+1,n]}) \text{ if } u \geq a+1,$$

and

$$C_{i,u}^{(a)} = (\tilde{D}_{[1,i-1]}^X, \mathbf{0}, D_{i,u+1}^{\prime*}, \dots, D_{i,u_{k_i}}^{\prime*}, D_{[i+1,n]}^Y) \text{ if } 1 \leq u \leq a,$$

we write

$$|\mathbb{E}((\tilde{X}_{ij}\tilde{X}_{il} - \mathbb{E}(X_{ij}X_{il}))\partial_{ij}\partial_{il}s(C_{i,u}))| \leq I_1 + I_2. \quad (40)$$

where

$$I_1 = |\mathbb{E}((\tilde{X}_{ij}\tilde{X}_{il} - \mathbb{E}(X_{ij}X_{il}))\partial_{ij}\partial_{il}(s_n(C_{i,u}) - s_n(C_{i,u}^{(a)})))|$$

and

$$I_2 = |\mathbb{E}((\tilde{X}_{ij}\tilde{X}_{il} - \mathbb{E}(X_{ij}X_{il}))\partial_{ij}\partial_{il}s(C_{i,u}^{(a)}))|.$$

By using the multivariate Taylor expansion of first order for $\partial_{ij}\partial_{il}s$, taking into account the definitions of $C_{i,u}$ and $C_{i,u}^{(a)}$ and then by using (14), we derive, after simple computations, that

$$I_1 \ll \frac{1}{n^{5/2}} \sum_{v=2}^{a+1} \sum_{r \in I'_v} \|(\tilde{X}_{ij}\tilde{X}_{il} - \mathbb{E}(X_{ij}X_{il}))(\tilde{X}_{ir} - \mathbb{E}(\tilde{X}_{ir}|\mathcal{F}_{u-v}^X))\|_1 \ll \frac{1}{n^{5/2}}(Aap)\sigma^2. \quad (41)$$

Next, using (14) again and the definition of the conditional expectation, we infer that

$$I_2 \ll \frac{1}{n^2} \|\mathbb{E}(\tilde{X}_{ij}\tilde{X}_{il}|\sigma(C_{i,u}^{(a)})) - \mathbb{E}(X_{ij}X_{il})\|_1.$$

Notice now that, since \mathbf{X}_n and \mathbf{Y}_n are assumed to be independent and since the rows of \mathbf{X}_n are independent, $\mathbb{E}(\tilde{X}_{ij}\tilde{X}_{il}|\sigma(C_{i,u}^{(a)})) = \mathbb{E}(\tilde{X}_{ij}\tilde{X}_{il}|\mathcal{F}_{i,u-a}^X)$. Therefore, after simple computations based on the definition of \tilde{X}_{ij} and on the fact that $A\|X_{ij}I(|X_{ij}| > A)\|_1 \leq \|X_{ij}^2I(|X_{ij}| > A)\|_1$, we obtain

$$I_2 \ll \frac{1}{n^2} \|\mathbb{E}(X_{ij}X_{il}|\mathcal{F}_{i,u-a}^X) - \mathbb{E}(X_{ij}X_{il})\|_1 + \frac{1}{n^2} \|X_{ij}I(|X_{ij}| > A)\|_2 \|X_{il}I(|X_{il}| > A)\|_2. \quad (42)$$

Starting from (39) and taking into account (40), (41) and (42), we get

$$T_1 \ll \frac{1}{n^{3/2}}(Aap^2)\sigma^2 + \frac{p}{n^2} \sum_{j=1}^i \|X_{ij}^2I(|X_{ij}| > A)\|_1 + \frac{1}{n^2} k_i p^2 \gamma_{aq}. \quad (43)$$

So, overall, starting now from the inequality (34), taking into account (35), (37), (38) and (43), and summing over i , we obtain that

$$|\mathbb{E}s_n(\tilde{D}^X) - \mathbb{E}s_n(D^Y)| \ll \frac{1}{n^{1/2}} p^2 \sigma^2 (A + aA + \sigma) + pL(A) + \eta_q^2 + p\gamma_{aq}. \quad (44)$$

Step 4: End of the proof.

Starting from (21), taking $A = \varepsilon\sqrt{n}$ and considering the upper bound (44), we get

$$\begin{aligned} |\mathbb{E}S^{\mathbf{X}_n}(z) - \mathbb{E}S^{\mathbf{Y}_n}(z)| &\ll p^2 \sigma^2 (\varepsilon + a\varepsilon + \frac{1}{n^{1/2}} \sigma) + pL(\varepsilon\sqrt{n}) + \eta_q^2 + p\gamma_{aq} \\ &\quad + \left(\frac{q}{p} + \frac{q+p}{n}\right)^{1/2} \sigma + L^{1/2}(\varepsilon\sqrt{n}) + \eta_q. \end{aligned}$$

Therefore, when $n \rightarrow \infty$, we obtain for all p, q, a , and ε ,

$$\limsup_{n \rightarrow \infty} |\mathbb{E}S^{\mathbf{X}_n}(z) - \mathbb{E}S^{\mathbf{Y}_n}(z)| \ll p^2 \sigma^2 (\varepsilon + a\varepsilon) + \eta_q^2 + \eta_q + p\gamma_{aq} + (q/p)^{1/2} \sigma.$$

Now we let $\varepsilon \rightarrow 0$ and obtain

$$\limsup_{n \rightarrow \infty} |\mathbb{E}S^{\mathbf{X}_n}(z) - \mathbb{E}S^{\mathbf{Y}_n}(z)| \ll \eta_q^2 + \eta_q + p\gamma_{aq} + (q/p)^{1/2} \sigma.$$

Then we let $a \rightarrow \infty$, and, by our hypotheses, for any p and q we obtain

$$\limsup_{n \rightarrow \infty} |\mathbb{E}S^{\mathbf{X}_n}(z) - \mathbb{E}S^{\mathbf{Y}_n}(z)| \ll \eta_q^2 + \eta_q + (q/p)^{1/2} \sigma.$$

Now we can let p and q tend to ∞ in such a way $q/p \rightarrow 0$ to obtain the desired result. \diamond

4.3 Proof of Corollary 8

By the reverse martingale convergence theorem and condition (3), we get that $\lim_{n \rightarrow \infty} \mathbb{E}(X_0 | \mathcal{G}_{-n}) = \mathbb{E}(X_0 | \mathcal{G}_{-\infty}) = 0$ a.s. So, since X_0 belongs to \mathbb{L}^2 , this last convergence implies that condition (8) holds. We prove now that under the conditions of the corollary, condition (9) is satisfied. Note first that, by stationarity, this latter condition reads as

$$\sup_u \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (45)$$

To prove that (45) holds we shall prove that

$$\lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{u \geq p+1} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 = 0, \quad (46)$$

and that

$$\lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq u \leq p} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 = 0. \quad (47)$$

To prove (46), we note that

$$\begin{aligned} \sup_{u \geq p+1} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 &\leq \sup_{u \geq p+1} \|\mathbb{E}(X_0 X_u | \mathcal{G}_0) - \mathbb{E}(X_0 X_u)\|_1 \\ &= \sup_{u \geq p+1} \|X_0 \mathbb{E}(X_u | \mathcal{G}_0) - \mathbb{E}(X_0 X_u)\|_1 \\ &\leq 2\|X_0\|_2 \cdot \sup_{u \geq p+1} \|\mathbb{E}(X_u | \mathcal{G}_0)\|_2 \leq 2\|X_0\|_2 \cdot \|\mathbb{E}(X_0 | \mathcal{G}_{-p})\|_2. \end{aligned}$$

This shows that (46) holds since (8) does under (3). We turn now to the proof of (47). By the reverse martingale convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \max_{1 \leq u \leq p} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 &= \max_{1 \leq u \leq p} \lim_{n \rightarrow \infty} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-n}) - \mathbb{E}(X_0 X_u)\|_1 \\ &= \sup_{1 \leq u \leq p} \|\mathbb{E}(X_0 X_u | \mathcal{G}_{-\infty}) - \mathbb{E}(X_0 X_u)\|_1, \end{aligned}$$

which is equal to zero by condition (4). This ends the proof of (47) and then of the corollary. \diamond

4.4 Proof of Theorem 9

It is well-known that for deriving the limiting spectral distribution of \mathbb{B}_N it is enough to study the Stieltjes transform of the following symmetric matrix of order $n = N + p$:

$$\mathbb{X}_n = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{0}_{p,p} & \mathcal{X}_{N,p}^T \\ \mathcal{X}_{N,p} & \mathbf{0}_{N,N} \end{pmatrix}.$$

Indeed the eigenvalues of \mathbb{X}_n^2 are the eigenvalues of $N^{-1} \mathcal{X}_{N,p}^T \mathcal{X}_{N,p}$ together with the eigenvalues of $N^{-1} \mathcal{X}_{N,p} \mathcal{X}_{N,p}^T$. Since these two latter matrices have the same nonzero eigenvalues, the following relation holds: for any $z \in \mathbb{C}^+$, $S_{\mathbb{B}_N}(z) = z^{-1/2} \frac{n}{2p} S_{\mathbb{X}_n}(z^{1/2}) + \frac{N-p}{2pz}$ (see, for instance, page 549 in Rashidi Far *et al.* [22] for additional arguments leading to the relation above. Obviously a similar equation holds for the Gram random matrix \mathbb{H}_N associated with $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$, namely: $S_{\mathbb{H}_N}(z) = z^{-1/2} \frac{n}{2p} S_{\mathbb{Y}_n}(z^{1/2}) + \frac{N-p}{2pz}$, where \mathbb{Y}_n is defined as \mathbb{X}_n but with $X_{\mathbf{u}}$ replaced by $Y_{\mathbf{u}}$. Therefore, in order to prove the theorem, it suffices to show that, for any $z \in \mathbb{C}^+$,

$$\lim_{N \rightarrow \infty} |S_{\mathbb{X}_n}(z) - \mathbb{E}(S_{\mathbb{Y}_n}(z))| = 0 \text{ a.s.} \quad (48)$$

Note now that $\mathbb{X}_n := n^{-1/2} [x_{ij}^{(n)}]_{i,j=1}^n$ where $x_{ij}^{(n)} = \sqrt{\frac{n}{N}} X_{i-p,j} \mathbf{1}_{i \geq p+1} \mathbf{1}_{1 \leq j \leq p}$ if $1 \leq j \leq i \leq n$, and $x_{ij}^{(n)} = x_{ji}^{(n)}$ if $1 \leq i < j \leq n$. Similarly we can write $\mathbb{Y}_n := n^{-1/2} [y_{ij}^{(n)}]_{i,j=1}^n$ where the $y_{ij}^{(n)}$'s are defined as the $x_{ij}^{(n)}$'s but with $X_{i-p,j}$ replaced by $Y_{i-p,j}$. The theorem then follows by applying Remark 7 of Theorem 5 to the matrices \mathbb{X}_n and \mathbb{Y}_n defined above. \diamond

4.5 Proof of Theorem 1

According to Theorem 9 and Theorem B.9. in Bai and Silverstein (2010), the proof of Theorem 1 is reduced to show that, for any $z \in \mathbb{C}^+$

$$\lim_{N \rightarrow \infty} \mathbb{E}(S^{\mathbb{H}_N}(z)) = S(z), \quad (49)$$

where \mathbb{H}_N is the Gram matrix associated with a Gaussian random field $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}}$ having the same covariance structure as $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}}$ and $S(z)$ is a Stieltjes transform of a measure F and satisfies equation (5). To prove the convergence above, we shall proceed in two steps. In the first step we shall prove that (49) holds under the additional assumption that the spectral density of $(X_k)_{k \in \mathbb{Z}}$ is square integrable. The proof of this particular case is facilitated by the fact that a square integrable spectral density allows us to use the celebrated Szegő-Trotter theorem for Toeplitz matrices. This assumption will be removed in a second step, where we approximate the spectral density by a square integrable one and then extend the characterization of the limit.

Step 1. Proof of (49) when the spectral density is square integrable.

We shall apply Theorem 1.1 in Silverstein (1995). Consider N independent copies $(g_{ij})_{j \in \mathbb{Z}}$, $i = 1, \dots, N$ of a sequence $(g_k)_{k \in \mathbb{Z}}$ of i.i.d. standard normal random variables. Set

$$\Gamma_p := \begin{pmatrix} c_0 & c_1 & \cdots & c_{p-1} \\ c_1 & c_0 & & c_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p-1} & c_{p-2} & \cdots & c_0 \end{pmatrix} \quad \text{where } c_k = \text{Cov}(X_0, X_k).$$

Using the stationarity of the Gaussian process $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$, we can easily verify that the random vector $((Y_{1j})_{1 \leq j \leq p}, \dots, (Y_{Nj})_{1 \leq j \leq p})$ has the same distribution as $(\mathbf{g}_1 \Gamma_p^{1/2}, \dots, \mathbf{g}_N \Gamma_p^{1/2})$ where for any $i \in \{1, \dots, N\}$, $\mathbf{g}_i = (g_{ij})_{1 \leq j \leq p}$ and $\Gamma_p^{1/2}$ is the symmetric non-negative square root of Γ_p . Therefore, for any $z \in \mathbb{C}^+$,

$$\mathbb{E}(S^{\mathbb{H}_N}(z)) = \mathbb{E}(S^{\Gamma_p^{1/2} \mathbb{G}_N \Gamma_p^{1/2}}(z)),$$

where $\mathbb{G}_N = \frac{1}{N} \mathcal{G}_{N,p}^T \mathcal{G}_{N,p}$ with $\mathcal{G}_{N,p} = (g_{ij})_{1 \leq i \leq N, 1 \leq j \leq p}$. Hence, according to Theorem 1.1 in Silverstein (1995), if $p/N \rightarrow c \in (0, \infty)$ and

$$F^{\Gamma_p} \text{ converges to a probability distribution } H \text{ as } p \rightarrow \infty, \quad (50)$$

then there is a nonrandom probability distribution F such that

$$d(F^{\mathbb{H}_N}, F) \rightarrow 0 \text{ a.s.} \quad (51)$$

Furthermore, the Stieltjes transform $S = S(z)$, $z \in \mathbb{C}^+$, of F satisfies the equation

$$S = \int \frac{1}{x(1 - c - czS) - z} dH(x).$$

Setting $\underline{S} := -(1 - c)/z + cS$, this last equation becomes

$$z = -\frac{1}{\underline{S}} + c \int \frac{x}{1 + x\underline{S}} dH(x). \quad (52)$$

We mention that \underline{S} is also a Stieltjes transform (see relation (1.3) in [23] or [15]), so $\text{Im } \underline{S} > 0$ for $z \in \mathbb{C}^+$.

Note now that, since the spectral density f is assumed to be square integrable, by Parseval's identity we have that $\sum_{k \in \mathbb{Z}} c_k^2 < \infty$. Therefore by a version of the Szegő's theorem for Toeplitz

forms (see page 72 of Trotter (1984)), the convergence (50) holds and we have, for any φ which is continuous and bounded,

$$\int \varphi(x) dH(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(2\pi f(\lambda)) d\lambda.$$

Since the function $\varphi(x) := x/(1+x\underline{S})$ is continuous and bounded by $1/\text{Im } \underline{S}$, the relation (52) can be rewritten as (5). To end the proof of (49) when the spectral density is assumed to be square integrable, it suffices to notice that (51) implies that $\lim_{N \rightarrow \infty} S^{\mathbb{H}_N}(z) = S(z)$ a.s. which in turn entails (49) since the Stieltjes transforms are bounded.

Step 2. Proof of (49) when the spectral density is not necessarily square integrable.

To remove the assumption on the square integrability of the spectral density, we shall truncate the spectral density, then define a Gaussian process with the help of the truncated spectral density. Next, we use the limit of the empirical eigenvalue distribution for this truncated process to approximate and then characterize the limit of $F^{\mathbb{H}_N}$.

In the rest of the proof, $(\xi_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^2}$ is a sequence of i.i.d. standard normal real-valued random variables. According to Proposition 15, there is no loss of generality by assuming from now on that $(Y_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}}$ has the following linear representation: for any k, ℓ in \mathbb{Z} ,

$$Y_{k\ell} = \sum_{j \in \mathbb{Z}} a_j \xi_{k, \ell-j} \quad \text{with} \quad a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ikx} \sqrt{f(x)} dx. \quad (53)$$

For a fixed positive real b , we define another centered real-valued Gaussian random field $(Z_{\mathbf{u}}^b)_{\mathbf{u} \in \mathbb{Z}^2}$ with the help of the function

$$f_b = f \wedge b.$$

Note that since f is a nonnegative, even and integrable function on $[-\pi, \pi]$, so is f_b . Then f_b is also the spectral density on $[-\pi, \pi]$ of a \mathbb{L}^2 -stationary process. Therefore, according to Proposition 15, if we set, for any k, ℓ in \mathbb{Z} ,

$$\tilde{a}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ikx} \sqrt{f_b(x)} dx \quad \text{and} \quad Z_{k\ell}^b = \sum_{j \in \mathbb{Z}} \tilde{a}_j \xi_{k, \ell-j}, \quad (54)$$

$(Z_{\mathbf{u}}^b)_{\mathbf{u} \in \mathbb{Z}^2}$ is a centered real-valued stationary Gaussian random field. In addition, for any fixed integer k , $(Z_{k\ell}^b)_{\ell \in \mathbb{Z}}$ admits f_b as spectral density on $[-\pi, \pi]$. Let \mathbb{H}_N^b be the Gram matrix associated with $(Z_{\mathbf{u}}^b)_{\mathbf{u} \in \mathbb{Z}^2}$. Since f_b is bounded, it is in particular square integrable. Then, by the Step 1 of the proof, we conclude that there is a nonrandom distribution function F^b such that

$$\lim_{N \rightarrow \infty} d(F^{\mathbb{H}_N^b}, F^b) = 0 \text{ a.s.} \quad (55)$$

On another hand, by using Lemma 12 together with Cauchy-Schwarz's inequality, we infer that

$$\mathbb{E} d^2(F^{\mathbb{H}_N}, F^b) \ll \frac{1}{Np} \left\| \sum_{i=1}^N \sum_{j=1}^p (Y_{ij}^2 + (Z_{ij}^b)^2) \right\|_1^{1/2} \left\| \sum_{i=1}^N \sum_{j=1}^p (Y_{ij} - Z_{ij}^b)^2 \right\|_1^{1/2}.$$

Since $\mathbb{E}(Y_{ij}^2) = \sum_{k \in \mathbb{Z}} a_k^2$, $\mathbb{E}((Z_{ij}^b)^2) = \sum_{k \in \mathbb{Z}} \tilde{a}_k^2$ and $\mathbb{E}((Y_{ij} - Z_{ij}^b)^2) = \sum_{k \in \mathbb{Z}} (a_k - \tilde{a}_k)^2$, by using Parseval's identity, it follows that

$$\begin{aligned} \mathbb{E} d^2(F^{\mathbb{H}_N}, F^b) &\ll \left(\int_{-\pi}^{\pi} f(x) dx + \int_{-\pi}^{\pi} f_b(x) dx \right)^{1/2} \left(\int_{-\pi}^{\pi} (f^{1/2}(x) - f_b^{1/2}(x))^2 dx \right)^{1/2} \\ &\ll \left(\int_{-\pi}^{\pi} f(x) dx \right)^{1/2} \left(\int_{-\pi}^{\pi} f I(f > b)(x) dx \right)^{1/2}. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem

$$\lim_{b \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} d^2(F^{\mathbb{H}_N}, F^b) = 0. \quad (56)$$

Since for any positive reals b and b' we have

$$d^2(F^{b'}, F^b) \leq 2\mathbb{E} d^2(F^{\mathbb{H}_N}, F^{b'}) + 2\mathbb{E} d^2(F^{\mathbb{H}_N}, F^b),$$

(56) implies that F^b is Cauchy. Taking into account that the space of distribution functions endowed with Lévy metric is complete, we conclude that there is a nonrandom distribution function F such that $\lim_{b \rightarrow \infty} d(F^b, F) = 0$ which, combined with (56), also gives $\lim_{N \rightarrow \infty} \mathbb{E} d(F^{\mathbb{H}_N}, F) = 0$. If we denote by S the Stieltjes transform of F and by S^b the Stieltjes transform of F^b , by the continuity theorem (see for instance Theorem B.9 in [2]), we obtain, for any $z \in \mathbb{C}^+$, the convergence of $S^b(z)$ to $S(z)$ and the convergence in probability of $S^{\mathbb{H}_N}(z)$ to $S(z)$. Since the Stieltjes transforms are bounded, we also have $\lim_{N \rightarrow \infty} \mathbb{E}(S^{\mathbb{H}_N}(z)) = S(z)$, which completes the proof of the convergence (49).

We shall prove now that $S(z)$ satisfies (5). We start from the equation satisfied by S^b which was found in Step 1, namely

$$z = -\frac{1}{\underline{S}^b} + c \int_{-\pi}^{\pi} \frac{f_b(x)}{1 + 2\pi f_b(x) \underline{S}^b} dx, \quad (57)$$

with $\underline{S}^b := -(1-c)/z + cS^b$. We note at this point that, and for any z in \mathbb{C}^+ , we also have $\underline{S}(z) = \lim_{b \rightarrow \infty} \underline{S}^b(z)$. It follows that $\underline{S} = -(1-c)/z + cS$, where \underline{S} is also a Stieltjes transform, implying that $\text{Im}(\underline{S})(z) > 0$. Therefore, for almost all x in $[-\pi, \pi]$,

$$\lim_{b \rightarrow \infty} \frac{f_b(x)}{1 + 2\pi f_b(x) \underline{S}^b} = \frac{f(x)}{1 + 2\pi f(x) \underline{S}}.$$

Also, for all b sufficiently large,

$$\left| \frac{f_b(x)}{1 + 2\pi f_b(x) \underline{S}^b} \right| \leq \frac{1}{2\pi \text{Im}(\underline{S}^b)} \leq \frac{1}{\text{Im}(\underline{S})}.$$

By the Lebesgue dominated convergence theorem, by passing to the limit when $b \rightarrow \infty$ in (57) we obtain that \underline{S} satisfies equation (5). \diamond

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